Formulas for HAMC

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This note explains the formulas needed by the Physics classes for the C++ Monte Carlo "hamc". First we explain some goals of the Monte Carlo, then we describe how to compute some basic physics quantities: 1) kinematic variables, 2) cross section 3) rates 4) asymmetry. Finally we show the formulas for the electromagnetic radiative corrections.

I GOALS OF HAMC

The "Hall A Monte Carlo" ("hamc") will be used to simulate the PREX experiment, as well as other experiments like HAPPEX-III and PVDIS. The goals of the simulation include the following :

- Compute rates, average asymmetries, and required size of the detector.
- Compute systematic errors due to backgrounds and pileup.
- Compute systematic errors in analyzing power due to the acceptance, collimation, and radiative corrections.
- Simulate the PVDIS trigger and deadtime effects.
- Compute the sensitivities of measurements to position, angle, energy.
- Compute the noise due to rastering the beam.
- Study strategies for transverse asymmetry measurements.
- Study the theoretical model dependence in PREX.
- Optimize the target thickness.

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As an example illustrating goal 1, see figures 1 and 2. These are measured momentum distribution and X-Y distribution of tracks in the HRS spectrometer. We want to be able to reproduce similar spectrum for expected running conditions and be able, for example, to deduce how large a detector we need.

II ELASTIC SCATTERING FROM LEAD

A Kinematics

First we define some notation. The beam energy is denoted by E and is in units of GeV. For elastic scattering at an angle θ from a target of mass M the outgoing electron's energy can be computed from conservation of 4-momentum, and is $E' = E/(1 + \frac{E}{M}(1 - \cos\theta))$. The 4-momentum transfer squared is $Q^2 = 2EE'(1 - \cos\theta)$. Some other constants which appear below are the fine structure constant $\alpha = \frac{1}{137}$, the Fermi constant $G_F = 1.6637 \times 10^{-5}\,\mathrm{GeV}^{-2}$, and the Weinberg angle $\sin^2\theta_W \sim 0.227$.

B Cross Section

The cross section for elastic scattering electrons with momentum transfer squared Q^2 is given by

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}_{\mathrm{Mott}} |F_p(Q^2)|^2 \tag{1}$$

where

$$F_p(Q^2) = \frac{1}{4\pi} \int d^3r j_0(Qr)\rho_p(r) \tag{2}$$

is the form factor for protons, related to the integral of the charge density $\rho_p(r)$, and $\frac{d\sigma}{d\Omega}_{\mathrm{Mott}}$ is the well-known Mott cross section for point-like scattering. In practical terms :

$$\frac{d\sigma}{d\Omega}_{\mathrm{Mott}} = \frac{(\alpha Z \frac{hc}{2\pi} \cos\frac{\theta}{2})^2}{400E^2 \sin^4\frac{\theta}{2}} \tag{3}$$

TABLE 1. ^{208}**Pb Form Factor Data**

Q^2 GeV^2	$\lvert F_p(Q^2)\rvert^2$
0.000873203	0.97034
0.00137167	0.88688
0.00198221	0.78059
0.00268438	0.66235
0.00351613	0.54224
0.0044337	0.42851
0.00548666	0.32706
0.00661961	0.24135
0.00789379	0.17262
0.00924213	0.1203
0.0107375	0.082532
0.0123013	0.056716
0.0140179	0.040004
0.015797	0.029682
0.0177348	0.023441
0.0197293	0.019509
0.0218883	0.016676
0.0240982	0.014236
0.0264784	0.011872
0.0289037	0.0095329
0.0315052	0.0073035
0.0341459	0.0053129
0.0368928	0.0036678

where $Z = 82$ for lead and $\alpha = \frac{1}{137}$, $\frac{hc}{2\pi} = 0.19738$ GeV-fm, and 1 fm $= 10^{-15}$m, and the energy E is in GeV. The result using this formula is in units of barns per steradians ($\frac{\text{barn}}{\text{str}}$). Note, the cross sections from Horowitz are in units of millibarns per steradians, so to compare them you have to divide his by 1000.

Some representative form factor data for ^{208}Pb is shown in table 1 and a plot is in fig 3. Note that for PREX, $Q^2 = 0.00786\text{GeV}^2$. The form factor may be considered to be a "picture" of the nucleus; microscopic theories of nuclei must predict this picture.

One can also define a form factor for neutrons

$$F_n(Q^2) = \frac{1}{4\pi}\int d^3r j_0(Qr)\rho_n(r) \tag{4}$$

C Rates

It's not necessarily in the scope of Physics classes to compute the rates, but I'll go ahead and explain how to do that here. The "rate" is how many electrons per second are seen scattering into the solid angle $d\Omega$ of the spectrometer, most of which show up in the detector, depending on how efficiently they are transported by the spectrometer.

Let T be the length of target seen by the spectrometer (by the geometry of acceptance it might not be the entire physical length). Let ρ be the density of target material ($\frac{g}{\text{cm}^3}$), let A be the atomic number, and σ is the shorthand for the differential cross section $\frac{d\sigma}{d\Omega}$ in barns/str. Let I be the beam current, in particles per second. Note that a beam current of 100 μA corresponds to 6.25×10^{14} particles per second. Then the rate seen in the solid angle $d\Omega$ is

$$\text{Rate} = I \times \sigma \times 0.602 \times d\Omega \times T \times \rho / A \qquad (5)$$

where the factor 0.602 arises due to Avogadro's number being used.

D Asymmetry

By scattering polarized electrons, one can measure the parity-violating asymmetry which arises from an interference between electromagnetic and weak scattering amplitudes.

$$A_{LR} = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L}, \qquad (6)$$

where $\sigma_{L(R)}$ is the cross section for the scattering of left(right) handed electrons. The approximate theoretical value is

$$A_{LR} = \frac{G_F Q^2}{4\pi\alpha\sqrt{2}} \left[1 - 4\sin^2\theta_W - \frac{F_n(Q^2)}{F_p(Q^2)} \right] \qquad (7)$$

Since $1 - 4\sin^2\theta_W \approx 0$, the asymmetry A_{LR} is very sensitive to the neutron form factor, which is the main idea of the PREX experiment.

A more exact expression for both the cross section and A_{LR} has been computed by Horowitz [1] and is the basis of the look-up table. For more about the theory and the experiment, see also [2].

To test the look-up table we can use equations 1 and 7 since they are approximately correct.

III RADIATIVE CORRECTIONS

The goal of the applying radiative corrections is to compute an energy loss ΔE for tracks in the events. This ΔE will vary statistically between events. The energy loss will interact with other parts of the simulation, i.e. the spectrometer acceptance model, to produce a distribution of particles in the detector.

We will use a probability distribution for this energy loss. In the initialization phase of the code, the probability distributions are computed and broken into cells 1,2,3,4... etc. (see fig 4). This produces an array $\Delta E[N]$ where N is the number of cells and ΔE is the energy loss. During the execution of the event loop, a random number generator produces the index "i" of the cell, and the $\Delta E[i]$ is looked up from the array. Using this method we can rapidly simulate the energy loss due to the three processes: 1) External Bremsstrahlung, 2) Internal Bremsstrahlung, and 3) Ionization.

A External Bremsstrahlung

From Mo and Tsai [3] we can compute the Bremsstrahlung radiative tail. The "external" Bremsstrahlung is photon emission from all the far-away nuclei which influences the electron as it pass through the matter. This excludes the actual nuclear scattering itself (that is explained in the next section).

We consider a small energy interval dE (in this context dE is a small interval, *not* an energy loss ΔE), the probability of finding an electron in this interval with initial energy E_0 and final energy E (hence $\Delta E = E_0 - E$) is

$$\text{Probability} = I_e(E_0, E, t)\, dE \tag{8}$$

where t is the radiation length of the target. For PREX, $t \sim 0.1$. The formula for I_e is equation A.3 in [3].

$$I_e(E_0, E, t) = bt(E_0 - E)^{-1}\, [\frac{E}{E_0} + \frac{3}{4}(\frac{E_0 - E}{E_0})^2]\, (\ln\frac{E_0}{E})^{bt} \tag{9}$$

where

$$b = \frac{4}{3}(1 + \frac{1}{9}[\frac{Z+1}{Z+\psi}][\ln(183Z^{-1/3})]^{-1}) \tag{10}$$

and

$$\psi = \ln(1440 Z^{-2/3})/\ln(183 Z^{-1/3}) \tag{11}$$

In the above equations Z is the number of protons in the nucleus, ($Z = 82$ for lead) and "ln" is the natural logarithm. Notice that $b \approx \frac{4}{3}$. Also, for computational ease you may want to use the math formula using the exponential function "e":

$$a^b = e^{b \ln a} \qquad \text{e.g., } 2^3 = 8 = e^{3 \ln 2} \tag{12}$$

B Internal Bremsstrahlung

Internal Bremsstrahlung is defined as energy loss due to photon emission resulting from the hard scattering of the electron from the nucleus. We assume there is only one such scattering in each event. In the "equivalent radiator" approximation, we can consider the process to be governed by the same formulas as in the previous section, but with the target thickness t replaced by

$$t_{\text{equiv}} = \frac{3\alpha}{4\pi}[\ln(\frac{Q^2}{m^2}) - 1] \tag{13}$$

Here, Q^2 is the 4-momentum transfer squared, m is the mass of the electron (in energy units, $m = 5.11 \times 10^{-4}$ GeV), and $\alpha = \frac{1}{137}$ as previously defined.

The effect of the hard scattering is to compute energy loss before and after scattering; the probability distributions in each case (before/after) are computed using t_{equiv} in place of t.

C Ionization

Energy loss due to ionization is caused by knock-out of electrons in the atoms in the target. There is a complex probability distribution called the Landau tail, but at high energies ($E \geq 1$ GeV), the distribution is fairly narrow compared to the Bremstrahlung tail, so we will neglect the width. Therefore, the ionization loss may be approximated by :

$$dE = \frac{dE}{dx} \times t \tag{14}$$

where t is the thickness of the target, now in units of g/cm^2. The ionization energy-loss parameter $\frac{dE}{dx}$ can be looked up for each target material. For example, for lead $\frac{dE}{dx} = 1.123$ MeV / (g / cm^2). For carbon the parameter is 1.745. For a compound target we may use a weighted average of the $\frac{dE}{dx}$ parameters. For testing the code, we can assume we only need this for the target.

IV MULTIPLE SCATTERING

When the electron passes through material, it will be deflected by many small-angle scatterings. We want to compute an angular shift $d\theta = \theta - \theta_0$ where θ_0 is the central angle. In the following, we'll take $\theta_0 = 0$.

The probability distribution for the scattering is a Gaussian distribution. For scattering in a plane it is

$$P(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma_\theta} \exp(-\frac{\theta^2}{2\sigma_\theta^2}) \tag{15}$$

where "exp" is the exponential function. Notice that the probability integrated over all angles is 1, as required. Using this distribution we need to compute $d\theta = \theta$ for each event at each material boundary.

The width of this distribution, in terms of the target thickness t (in radiation lengths) is

$$\sigma_\theta = \frac{0.0136}{E}\, z\,\sqrt{t}\;(1 + 0.038\ln(t)) \tag{16}$$

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Here, z is the charge number of the electron ($z = 1$) and E is it's energy in GeV. For a compound target we may use a weighted average of t.

For testing the multiple scattering code, we can assume we only need it for the target. Eventually we'll need to setup scattering for each material boundary in the simulation.

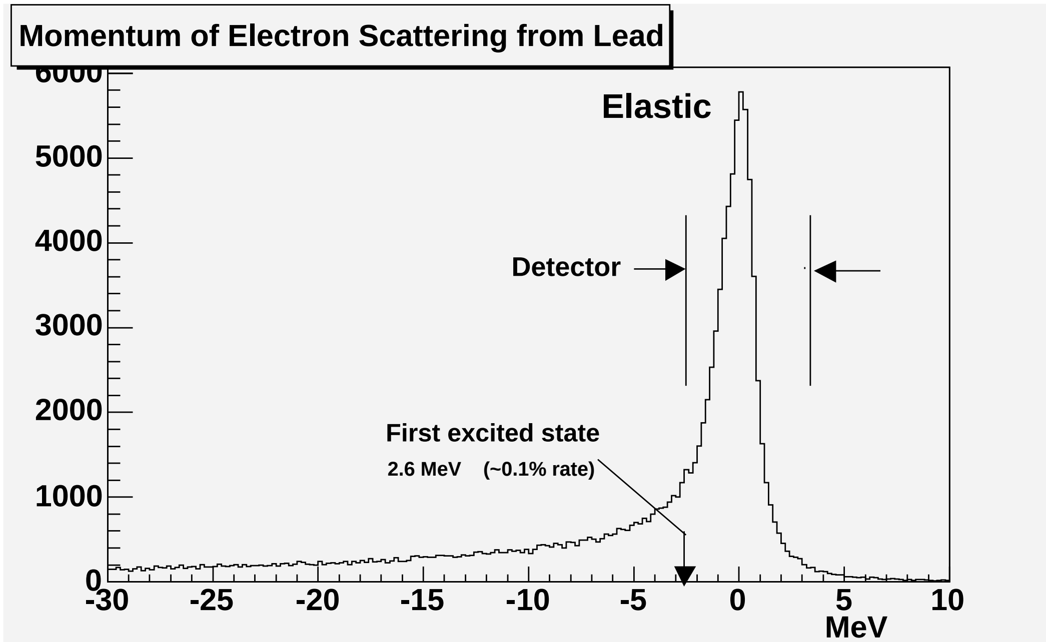

FIGURE 1. Momentum spectrum for elastic scattering from a 0.5 mm thick lead target at $E = 1.1$ GeV and $\theta = 6°$. Data taken in Nov 2005 with old septum magnet.

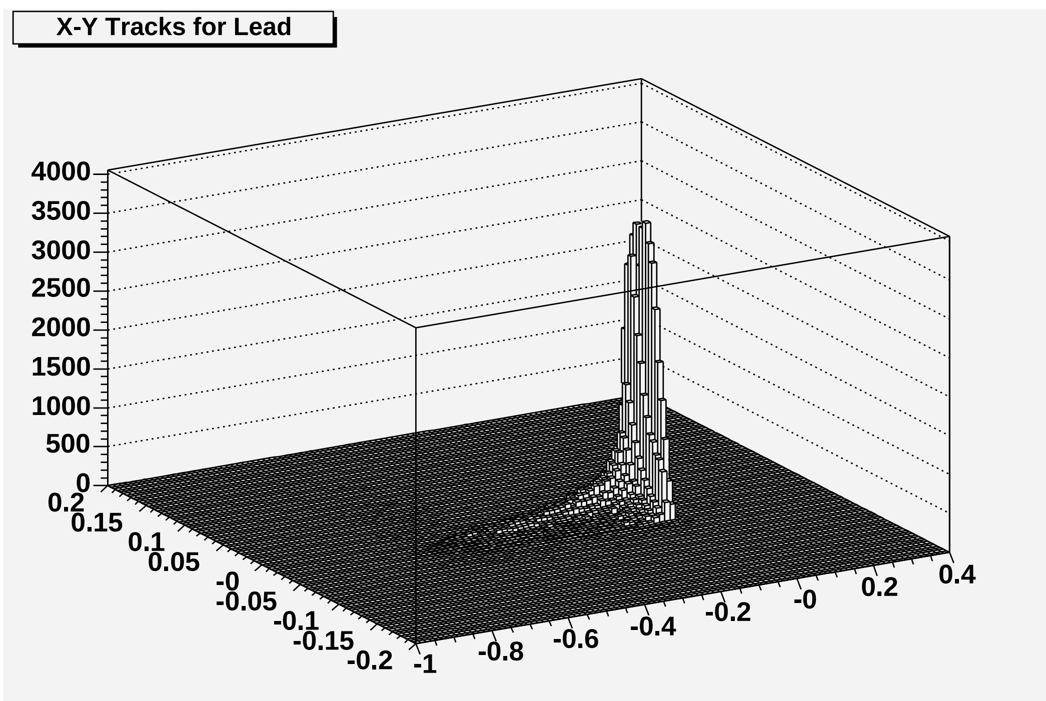

FIGURE 2. X-Y positions (meters) of tracks in HRS spectrometer, under conditions listed with fig 1.

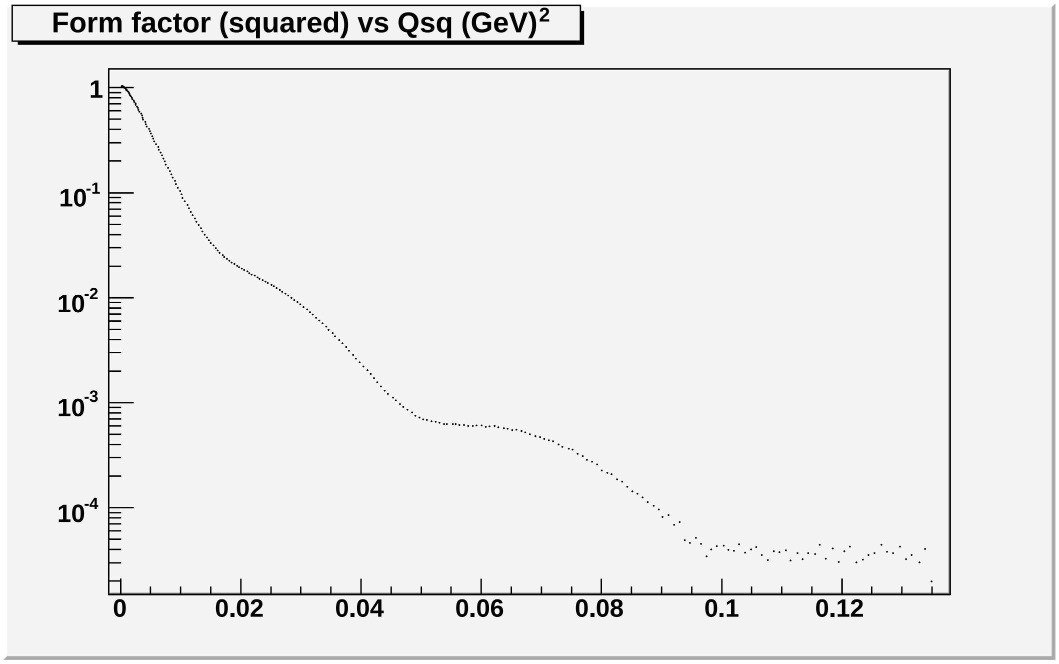

FIGURE 3. Form factor squared versus Q^2 for lead. For PREX, $Q^2 = 0.00786\text{GeV}^2$.

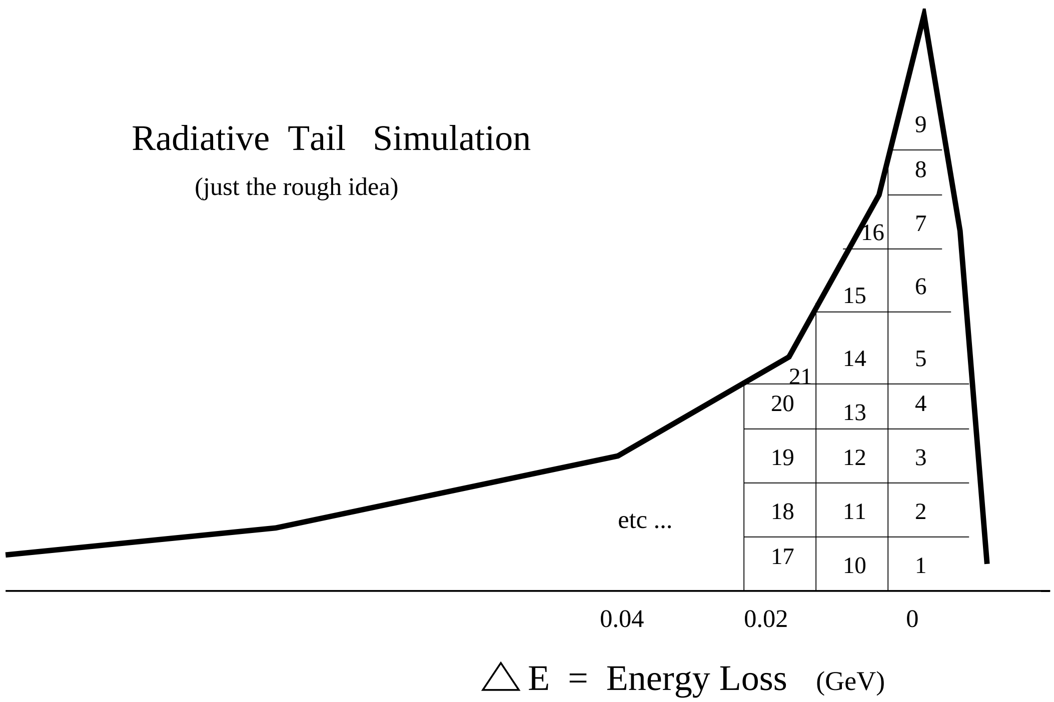

FIGURE 4. Strategy for simulating a radiative tail, or any other distribution. See text for details. Note, for radiative tail a test will be that the result should agree with the real data, for example fig 1, within the limits of the resolution (smearing) of the measurement system.

REFERENCES

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3. L.W. Mo and Y.S.Tsai, Rev. Mod. Phys **41** 205 (1969).