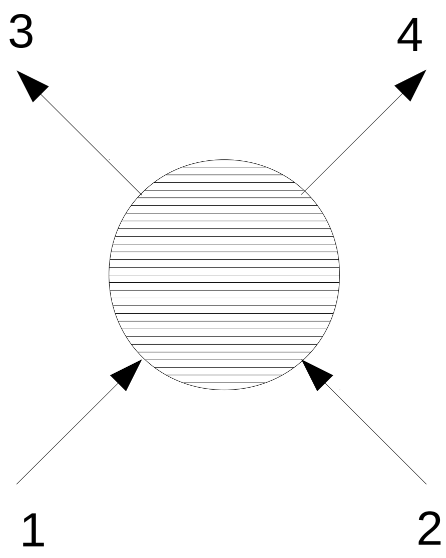
Two Body Scattering and Mandelstam Variables

$$1(k_1) + 2(p_1) \to 3(k_2) + 4(p_2)$$



- Equivalence principle requires observable to be expressed in terms of Lorentz scalers
- 10 Lorentz scalers can be constructed out of 4 Lorentz vectors
- Constraints: Four-momentum conservation and all 4 particles are on-shell

Two independent kinematic variables for unpolarized two-body scattering process

Mandelstam variables are convenient choices

$$s = (k_1 + p_1)^2 = (k_2 + p_2)^2$$

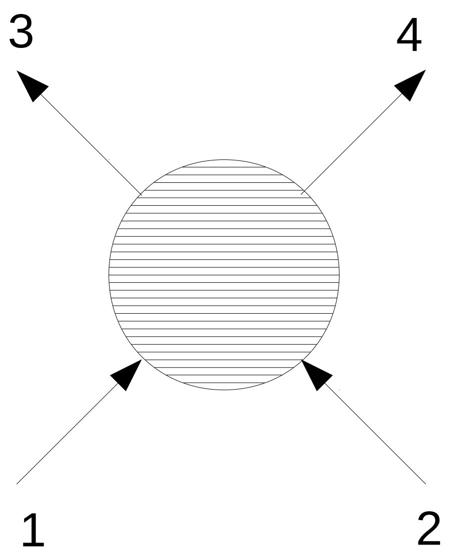
$$t = (k_1 - k_2)^2 = (p_2 - p_1)^2$$

$$u = (k_1 - p_2)^2 = (p_1 - k_2)^2$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Two Body Scattering and Mandelstam Variables

$$1(k1) + 2(P1) \rightarrow 3(k2) + 4(P2)$$



Conservation of four-momentum leads to 3 independent vectors:

$$K = \frac{k_1 + k_2}{2} \qquad P = \frac{p_1 + p_2}{2}$$

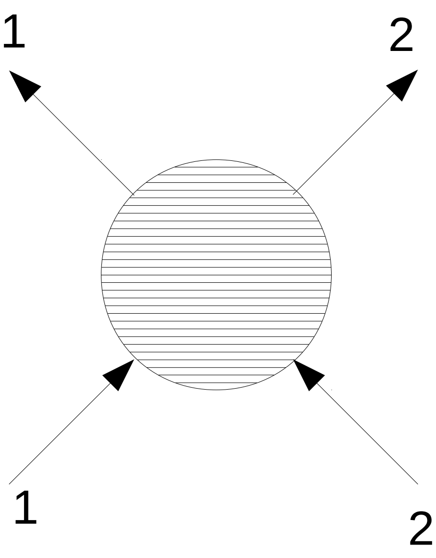
$$q = k_1 - k_2 = p_2 - p_1$$

Some relation between these three vectors and Mandelstam variables:

$$q \cdot K = q \cdot \ P = 0 \qquad t = q^2 \qquad P \cdot K = \frac{s - u}{4}$$

Elastic Scattering of Two Dirac Particles

$$1(k) + 2(p) \to 1(k') + 2(p')$$



Scattering amplitude can be expressed as:

$$M = \bar{u}(k')\Gamma_1 u(k) \cdot \bar{u}(p')\Gamma_2 u(p)$$

- Γ_1 and Γ_2 are 4x4 matrices, and can be expressed in terms of Dirac γ-matrices:

$$1 \qquad \gamma^\mu \qquad \gamma^5 \qquad \gamma^\mu\gamma^5 \qquad \sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

- Dirac spinors satisfy the Dirac equation:

$$\bar{u}(\gamma^\mu p_\mu - m) = 0 \qquad (\gamma^\mu p_\mu - m)u = 0$$

Construction of Scattering Amplitude

First we consider vector current:

$$\bar{u}(k')\Gamma^{\rho}_{1V}u(k) = \bar{u}(k')\left[\gamma^{\rho}A_1 + i\sigma^{\rho\lambda}q_{\lambda}A_2 + A_3 q^{\rho} + i\sigma^{\rho\lambda}K_{\lambda}A_4 + A_5 K^{\rho}\right]u(k)$$

A1, …, A5 are functions of the two Mandelstam variables s-u and t

- This is the most general construction of the vector current in terms of Dirac spinors, Dirac γ-matrices and the four-vectors of incident and outgoing particles
- The five terms in square brackets are not independent

It can be proved from the Dirac equation:

$$\bar{u}(k')K^{\rho}u(k) = \bar{u}(k')\left[\frac{i}{2}\sigma^{\rho\lambda}q_{\lambda} + m_1\gamma^{\rho}\right]u(k)$$

$$i\bar{u}(k')\sigma^{\rho\lambda}K_{\lambda}u(k) = \frac{1}{2}\bar{u}(k')q^{\rho}u(k)$$

Thus, the general form of a vector current of particle 1 can be expressed as:

$$\boxed{\bar{u}(k')\Gamma^{\rho}_{1V}u(k) = \bar{u}(k')\left[\gamma^{\rho}G_1 + G_2K^{\rho} + G_3q^{\rho}\right]u(k)}$$

Construction of Scattering Amplitude

General form for vector current of particle 1:

$$\bar{u}(k')\Gamma^{\rho}_{1V}u(k) = \bar{u}(k')\left[\gamma^{\rho}G_1 + G_2K^{\rho} + G_3q^{\rho}\right]u(k)$$

Similarly, the axial vector current can be express as:

$$\bar{u}(k')\Gamma^{\rho}_{1A}u(k) = \bar{u}(k')\left[\gamma^{\rho}\gamma^5H_1 + H_2K^{\rho}\gamma^5 + H_3q^{\rho}\gamma^5\right]u(k)$$

And for particle 2, we have

$$\bar{u}(p')\Gamma^{\rho}_{2V}u(p) = \bar{u}(p')\left[\gamma^{\rho}g_1 + g_2P^{\rho} + g_3q^{\rho}\right]u(p)$$

$$\bar{u}(p')\Gamma^{\rho}_{2A}u(p) = \bar{u}(p')\left[\gamma^{\rho}\gamma^5h_1 + h_2P^{\rho}\gamma^5 + h_3q^{\rho}\gamma^5\right]u(p)$$

- The “form factors” G_1, G_2, G_3, H_1, H_2, H_3, g_1, g_2, g_3, h_1, h_2, h_3 are functions of the Mandelstam variables t, s-u

Construction of Scattering Amplitude

Scattering amplitude:

$$M = \bar{u}(k')\Gamma_1 u(k) \cdot \bar{u}(p')\Gamma_2 u(p) = \bar{u}(k')(\Gamma_{1V} + \Gamma_{1A})u(k) \cdot \bar{u}(p')(\Gamma_{2V} + \Gamma_{2A})u(p)$$

Parity properties:

Vector * Vector	+1
Vector * Axial vector	-1
Axial vector * Vector	-1
Axial vector * Axial vector	+1

- Parity conservation prohibits mixture of terms with different parity property in total amplitude
- Leading-order QED contains only vector vertex, total amplitude has positive parity

✔ We only need to consider terms with *positive* parity in the total amplitude.

$$\boxed{M = \bar{u}(k')\Gamma_{1V} u(k) \cdot \bar{u}(p')\Gamma_{2V} u(p) + \bar{u}(k')\Gamma_{1A} u(k) \cdot \bar{u}(p')\Gamma_{2A} u(p)}$$

Construction of Scattering Amplitude

Vector-vector coupling:

$$\bar{u}(k')\Gamma^{\rho}_{1V}u(k) = \bar{u}(k')\left[\gamma^{\rho}G_1 + G_2K^{\rho} + G_3q^{\rho}\right]u(k)$$

$$\bar{u}(p')\Gamma^{\rho}_{2V}u(p) = \bar{u}(p')\left[\gamma^{\rho}g_1 + g_2P^{\rho} + g_3q^{\rho}\right]u(p)$$

Take only the terms with *positive charge parity* in the product:

$$\bar{u}(k')\Gamma_{1V}u(k)\bar{u}(p')\Gamma_{2V}u(p) = g_1G_1\bar{u}(k')\gamma^{\rho}u(k)\bar{u}(p')\gamma_{\rho}u(p)$$
$$+\left(g_2G_2K\cdot P + g_3G_2K\cdot q + G_3g_2P\cdot q + g_3G_3q^2\right)\bar{u}(k')u(k)\bar{u}(p')u(p)$$

Remember: $q\cdot K = q\cdot\ P = 0$

$$\bar{u}(k')\Gamma_{1V}u(k)\bar{u}(p')\Gamma_{2V}u(p) = g_1G_1\bar{u}(k')\gamma^{\rho}u(k)\bar{u}(p')\gamma_{\rho}u(p)$$
$$+\left(g_2G_2K\cdot P + g_3G_3q^2\right)\bar{u}(k')u(k)\bar{u}(p')u(p)$$

$$\boxed{\bar{u}(k')\Gamma_{1V}u(k)\bar{u}(p')\Gamma_{2V}u(p) = F_1\bar{u}(k')\gamma^{\rho}u(k)\bar{u}(p')\gamma_{\rho}u(p) + F_2\bar{u}(k')u(k)\bar{u}(p')u(p)}$$

Construction of Scattering Amplitude

Vector-vector coupling:

$$\bar{u}(k')\Gamma_{1V}u(k)\bar{u}(p')\Gamma_{2V}u(p) = F_1\bar{u}(k')\gamma^\rho u(k)\bar{u}(p')\gamma_\rho u(p) + F_2\bar{u}(k')u(k)\bar{u}(p')u(p)$$

Similarly, for axial vector-axial vector coupling, we have

$$\begin{aligned}\bar{u}(k')\Gamma_{1A}u(k)\bar{u}(p')\Gamma_{2A}u(p) =& F_3\bar{u}(k')\gamma^\rho\gamma^5 u(k)\bar{u}(p')\gamma_\rho\gamma^5 u(p) + F_4\bar{u}(k')\gamma\cdot P\gamma^5 u(k)\bar{u}(p')\gamma^5 u(p)\\ &+ F_5\bar{u}(k')\gamma^5 u(k)\bar{u}(p')\gamma\cdot K\gamma^5 u(p) + F_6\bar{u}(k')\gamma^5 u(k)\bar{u}(p')\gamma^5 u(p)\end{aligned}$$

- In total we have six “form factors”
- The six “form factors” depend on two independent kinematic variables s-u and t

ep Scattering Beyond the OPE approximation

ep scattering beyond the Born approximation:

$$M=\frac{4\pi\alpha}{Q^2}\bar{u}'\gamma^{\mu}u\cdot\bar{N}'\left(\tilde{F}_1\gamma^{\mu}-\tilde{F}_2[\gamma^{\mu},\gamma^{\nu}]\frac{q_{\nu}}{4M}+\tilde{F}_3K_{\nu}\gamma^{\nu}\frac{P^{\mu}}{M^2}\right)N$$

$$\tilde{G}_E=\tilde{F}_1-\tau\tilde{F}_2$$

$$\tilde{G}_M=\tilde{F}_1+\tilde{F}_2$$

Born contribution + 1γ×2γ interference

- The three complex amplitudes depend on both Q^2 and ε
- They go to the Born form factor GE, GM, and 0 respectively in the one-photon approximation
- The deviations from the Born form factors are of the same of the fine-structure constant

Observables beyond the Born approximation:

Rosenbluth separation method:

$$\sigma_R \simeq \frac{|\widetilde{G}_M|^2}{\tau}\left[\tau + \varepsilon \frac{|\widetilde{G}_E|^2}{|\widetilde{G}_M|^2} + 2\varepsilon\left(\tau + \frac{|\widetilde{G}_E|}{|\widetilde{G}_M|}\right)\Re\left(\frac{\nu \widetilde{F}_3}{M^2|\widetilde{G}_M|}\right)\right]$$

$$\left(R^{\exp}_{Rosenbluth}\right)^2 = \frac{|\widetilde{G}_E|^2}{|\widetilde{G}_M|^2} + 2\left(\tau + \frac{|\widetilde{G}_E|}{|\widetilde{G}_M|}\right) Y_{2\gamma}$$

Polarization transfer technique:

$$\frac{P_t}{P_l} \simeq -\sqrt{\frac{2\varepsilon}{\tau(1+\varepsilon)}}\left[\frac{|\widetilde{G}_E|}{|\widetilde{G}_M|} + \left(1 - \frac{2\varepsilon}{1+\varepsilon}\right)\frac{|\widetilde{G}_E|}{|\widetilde{G}_M|}\Re\left(\frac{\nu \widetilde{F}_3}{M^2|\widetilde{G}_M|}\right)\right]$$

$$R^{\exp}_{polarization} = \frac{|\widetilde{G}_E|}{|\widetilde{G}_M|} + \left(1 - \frac{2\epsilon}{1+\epsilon}\frac{|\widetilde{G}_E|}{|\widetilde{G}_M|}\right) Y_{2\gamma}$$

$$Y_{2\gamma}(\epsilon, Q^2) = \Re\left(\frac{\nu \widetilde{F}_3}{M^2|\widetilde{G}_M|}\right) \sim \alpha \simeq 1/137$$

$$\nu = M^2\sqrt{(1+\epsilon)/(1-\epsilon)}\sqrt{\tau(1+\tau)}$$

Guichon & Vanderhaeghen, Phys. Rev. Lett. 91 (2003) 142303

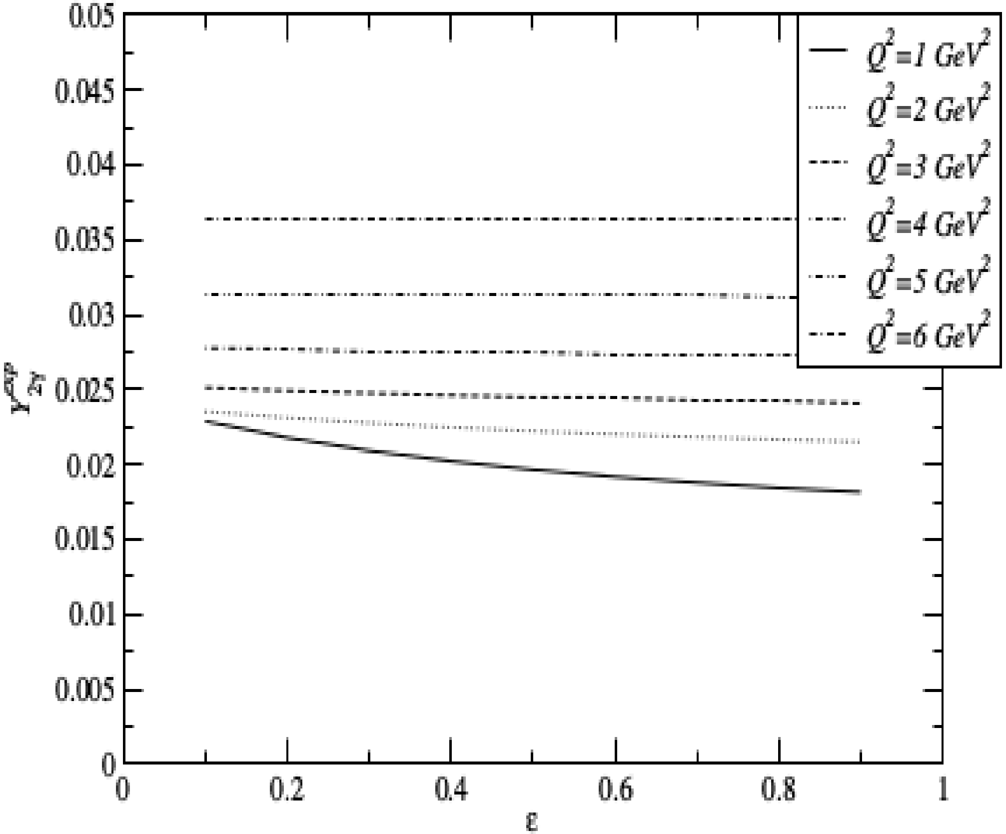
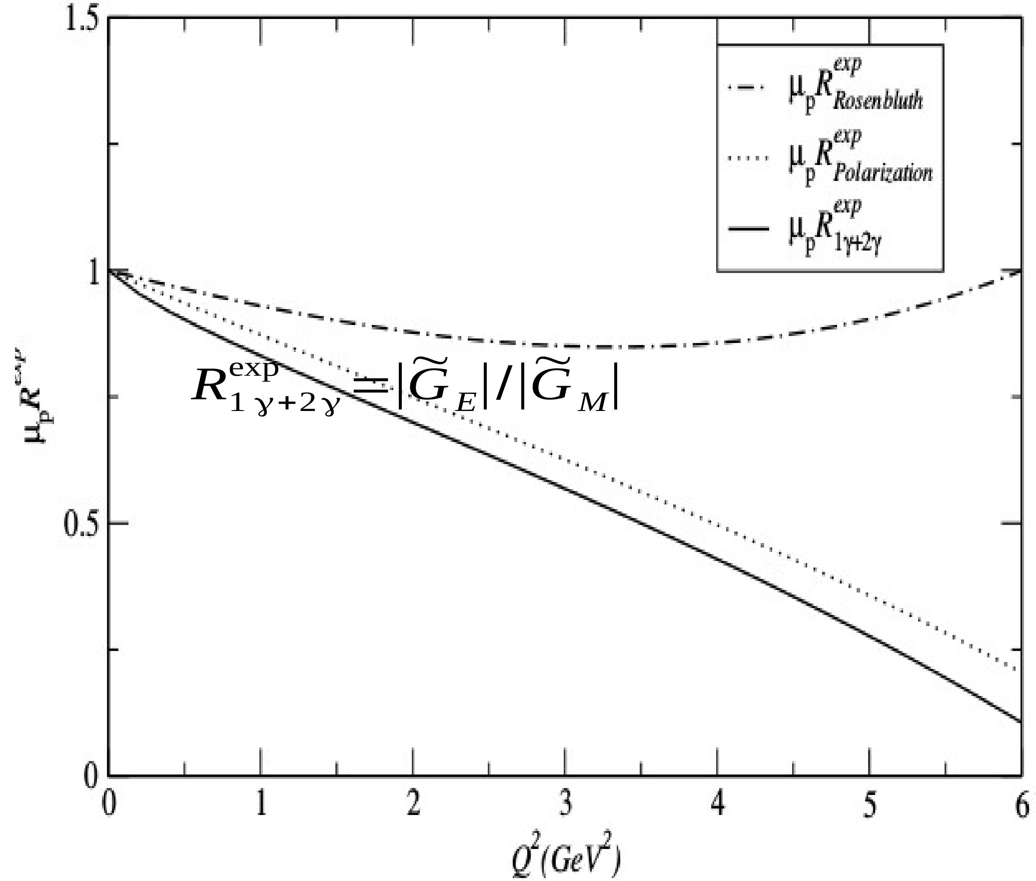


FIG. 3. The ratio $Y_{2\gamma}^{\rm exp}$ versus ε for several values of Q^2.



At $Q^2=3(\text{GeV})^2$, we get from the polynomial fit given in the paper:

$$\mu_p R^{\text{exp}}_{\text{Rosenbluth}} = 0.625$$

$$\mu_p R^{\text{exp}}_{\text{Rosenbluth}} = 0.851$$

Solve the system of equations for $|\tilde{G}_E|/|\tilde{G}_M|$ and $Y_{2\gamma}$, I get

$$\mu_p|\tilde{G}_E|/|\tilde{G}_M| = 0.56$$

$$Y_{2\gamma} = 0.0249$$