Two or three ε values for RLT separation?

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Abstract

In order to perform the RLT separation in E89-044, one could think that three different values of ε should be better than only two. The goal of the following study is to compare the two cases for all the chosen two body break up kinematics. The conclusion is unambiguous: a third point is never necessary.

1 Introduction

The total time allocated to each separation is fixed, but one needs to determine which fraction of this time has to be spent on each kinematics (ε_1 corresponding to two forward measurements, ε_3 to the backward measurement and a possible intermediate value ε_4 for a supplementary measurement). The criteria chosen to determine these time fractions is to minimize the error on the response functions σ_T, σ_{LT} and $\sigma_L + \sigma_{TT}$.

The measured cross sections S_i may be written in the following form: $S_i = \Gamma_i \times \Sigma_i$, where i denotes the kinematics index for the different settings, Γ_i is the virtual photon flux and Σ_i is a linear combination of the response functions. For instance, for two forward measurements (Σ_1 and Σ_2 with ε_1) and one backward measurement (Σ_3 with ε_3) in coplanar kinematics, we can write the following equations:

$$\Sigma_1 = \sigma_{\rm T} + \varepsilon_1 \left(\sigma_{\rm L} + \sigma_{\rm TT}\right) - \varepsilon_1' \,\sigma_{\rm LT} \tag{1}$$

$$\Sigma_2 = \sigma_{\rm T} + \varepsilon_1 \left(\sigma_{\rm L} + \sigma_{\rm TT}\right) + \varepsilon_1' \,\sigma_{\rm LT} \tag{2}$$

$$\Sigma_3 = \sigma_{\rm T} + \varepsilon_3 \left(\sigma_{\rm L} + \sigma_{\rm TT}\right) - \varepsilon_3' \,\sigma_{\rm LT} \tag{3}$$

where $\varepsilon_i = \left[1 + \frac{2\,{\rm q}^2}{{\rm Q}^2}\tan^2(\frac{\theta}{2})\right]^{-1}$ is the virtual photon linear polarization and $\varepsilon_i' = \sqrt{\varepsilon_{\rm i}\,(\varepsilon_{\rm i} + 1)}$.

When solving this set of equations, one obtains the response functions in terms of the Σ_i's:

$$\sigma_{\rm LT} = \frac{1}{2\,\varepsilon_1'}[\Sigma_2 - \Sigma_1] \tag{4}$$

$$\sigma_{\rm T} = \frac{1}{2\,\varepsilon_1'\,(\varepsilon_3 - \varepsilon_1)}\,[\Sigma_1\,(\varepsilon_1'\varepsilon_3 + \varepsilon_1\varepsilon_3') + \Sigma_3\,(\varepsilon_1'\varepsilon_3 - \varepsilon_1\varepsilon_3') - 2\,\varepsilon_1\varepsilon_1'\,\Sigma_3] \tag{5}$$

$$\sigma_{\rm L} + \sigma_{\rm TT} = \frac{1}{2\,\varepsilon_1'\,(\varepsilon_3 - \varepsilon_1)}\,[-\Sigma_1\,(\varepsilon_3' + \varepsilon_1') + \Sigma_2\,(\varepsilon_3' - \varepsilon_1') + 2\,\varepsilon_1'\Sigma_3] \tag{6}$$

Equation (4) gives the value of σ_{LT} and the error on this response function is given by:

$$\delta\sigma_{\rm LT} = \frac{1}{2\,\varepsilon_1'}\sqrt{(\delta\Sigma_1)^2 + (\delta\Sigma_2)^2} \tag{7}$$

Now, one needs to complete the RLT separation and to determine the other response functions σ_T and $\sigma_L + \sigma_{TT}$. The goal is to obtain these response functions with the lowest possible uncertainty considering the total time allocated to the measurement. *A priori*, it is not obvious that only two ε values are sufficient to perform the RLT separation with a high enough accuracy. Therefore, we shall now compare two different ways of doing this separation, the first one uses two different ε and the second one three of them.

The principle of these two methods is the following:

- first method: using two ε values and equations (5) and (6).
- second method: performing another measurement for an ε_4 intermediate between ε_1 and ε_3. This leads to a supplementary equation:

$$\Sigma_4 = \sigma_{\rm T} + \varepsilon_4\,(\sigma_{\rm L} + \sigma_{\rm TT}) - \varepsilon_4'\,\sigma_{\rm LT} \tag{8}$$

  which can be used in order to improve the accuracy of the RLT separation.

2 Optimization procedure with two ε values

In this method, the Fortran code used to determine the times allocated to each measurements is EPERP [2]. As input, it takes the cross sections, the virtual photon polarizations and ε values computed by the program HE3EEX [1].

Below, we shall explain the method in the case of a quasi-perpendicular kinematics because in the parallel one, the two forward measurements reduce to a single measurement (parallel kinematics is indeed a special case of the quasi-perpendicular one).

Let us assume that T_{lim} is the total time allocated to the set of three different settings necessary to perform an RLT separation. This time is divided between these three settings in T_1, T_2 and T_3 such that: $T_{lim} = T_1 + T_2 + T_3$. For computation purposes, we rewrite this formula as: $T_{lim} = T_1(1 + \alpha + \gamma)$ with $\alpha = \frac{T_2}{T_1}$ and $\gamma = \frac{T_3}{T_1}$.

As we have seen previously from equation (4), we determine σ_{LT}; this equation depends only on Σ_1 and Σ_2, therefore, the error $\delta\sigma_{LT}$ will only depends on the times T_1 and T_2. For the times T_1 and T_2 such that $T_1 + T_2 = T$ is fixed, $T_1 = \alpha T_2$, $T_1 = \frac{T}{1+\alpha}$ and $T_2 = \frac{\alpha T}{1+\alpha}$, $\delta\sigma_{LT}$ may be written as a function of T and α:

$$\delta\sigma_{LT} = \frac{1}{2\,\varepsilon_1'\sqrt{T}}\sqrt{\left[\Sigma_1^2 C_1(1+\alpha) + \Sigma_2^2 C_2 \frac{1+\alpha}{\alpha}\right]} \tag{9}$$

where $\frac{C_i}{T_i} = (\frac{\delta\Sigma_i}{\Sigma_i})^2$ is the statistical error for a time T_i. One just needs to find out the α parameter which minimizes this error by derivating equation 9 w.r.t. α:

$$\alpha = \frac{\Sigma_2\sqrt{C_2}}{\Sigma_1\sqrt{C_1}} \tag{10}$$

Now that α is fixed, we shall determine γ. This second parameter depends on T_3 which enters in both the error on σ_T and $\sigma_L + \sigma_{TT}$ (that we shall write σ_{LTT} in order to simplify the notation). We cannot simultaneously minimize $\delta\sigma_T$ and $\delta\sigma_{LTT}$, but we know that the error $\delta\sigma_{LTT}$ will be higher than $\delta\sigma_T$, we thus choose to minimize the following quantity:

$$\left[\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}\right]^2 = \frac{(1+\alpha+\gamma)(\varepsilon_1'\varepsilon_3 + \varepsilon_1\varepsilon_3')^2\,\Sigma_1^2\,C_1 \,+\, (\varepsilon_1'\varepsilon_3 - \varepsilon_1\varepsilon_3')^2\,\Sigma_3^2\,\frac{C_2}{\alpha} \,+\, 4\,\varepsilon_1'^2\,\Sigma_3^2\,\frac{C_3}{\gamma}}{T_{lim}(1+\alpha+\gamma)[-\Sigma_1\,(\varepsilon_1' \,+\, \varepsilon_3') \,+\, \Sigma_2(\varepsilon_3' \,-\, \varepsilon_1') \,+\, 2\,\varepsilon_1'\,\Sigma_3]^2} \tag{11}$$

Now, one needs to look at the minimum by derivating with respect to γ:

$$\gamma = \sqrt{\frac{4\,(\varepsilon_1'\,\Sigma_3)^2\,(1\,+\,\alpha)\,\mathrm{C}_3}{[(\varepsilon_1' \,+\, \varepsilon_3')\,\Sigma_1]^2\,\mathrm{C}_1 \,+\, [(\varepsilon_3' \,-\, \varepsilon_1')\,\Sigma_2]^2\,\frac{\mathrm{C}_2}{\alpha}}} \tag{12}$$

The two parameters are now determined and the times T_1, T_2 and T_3 also.

On figure 1, we show the variation of $\frac{\delta\sigma_{LT}}{\sigma_{LT}}$ versus α for a fixed $T_1 + T_2$ and for the α corresponding to the minimum, the variation of $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ versus γ. The values of α and γ given by the program EPERP correspond to the minima of the curves.

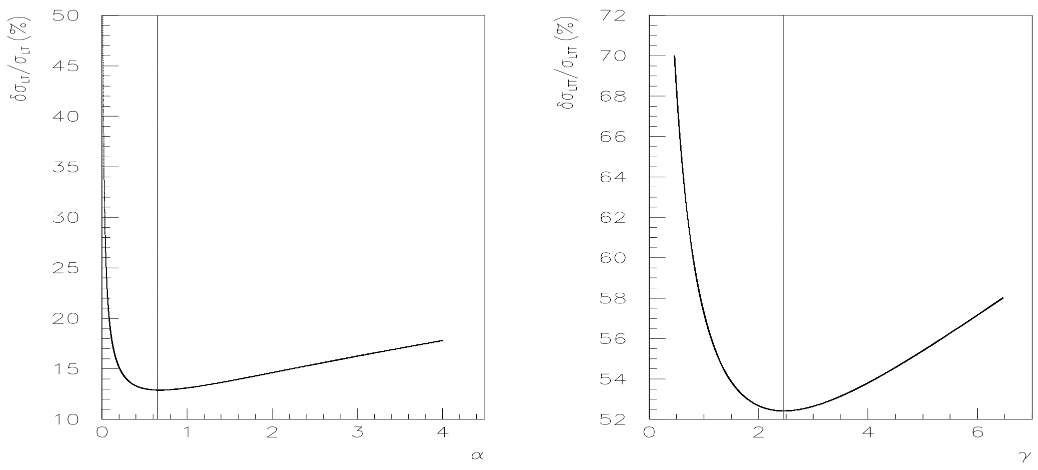

Figure 1: Left: variation of the relative error on σ_{LT} versus α. The blue line corresponds to the α value given by EPERP [2]. Right: variation of the relative error on σ_{LTT} versus γ. The blue line corresponds to the value given back by EPERP [2]. These results have been obtained for a quasi-perpendicular kinematics with p_m =300 MeV, ε_1 =0.6808 and ε_3 =2.4624

3 Optimization procedure with three ε values

The method using three ε values does not work in the same way as the EPERP program. It seemed less time consuming to pick up some random values for the sum T_1+T_2 within the interval [0,T_{lim}]. Once this sum was chosen, we determined, as previously, the α parameter that minimizes the error on σ_{LT} and therefore, this fixes the times T_1 and T_2. The second step of the procedure consists in picking up some random value for T_3 in the remaining amount of time $(T_{lim} - (T_1 + T_2))$; this fixes simultaneously the time T_4 allocated to the third ε value (ε_4). Equations (1), (2), (3) and (8) then lead to the following set of three equations with two unknowns:

$$\frac{\Sigma_1 + \Sigma_2}{2} = \sigma_{\mathrm{T}} + \varepsilon_1\,\sigma_{\mathrm{LTT}} \tag{13}$$

$$\Sigma_4 + \varepsilon_4'\,\sigma_{\mathrm{LT}} = \sigma_{\mathrm{T}} + \varepsilon_4\,\sigma_{\mathrm{LTT}} \tag{14}$$

$$\Sigma_3 + \varepsilon_3'\,\sigma_{\mathrm{LT}} = \sigma_{\mathrm{T}} + \varepsilon_3\,\sigma_{\mathrm{LTT}} \tag{15}$$

Assuming that ε_1, ε_3 and ε_4 are perfectly known and that we can determine the errors on Σ_1, Σ_2, Σ_3, Σ_4 and σ_{LT}, then the problem is equivalent to a linear regression.

In order to simplify the notation, let us consider the following variables:

$$y_1 = \frac{\Sigma_1 + \Sigma_2}{2} \tag{16}$$
$$y_2 = \Sigma_4 + \varepsilon'_4 \, \sigma_{LT} \tag{17}$$
$$y_3 = \Sigma_3 + \varepsilon'_3 \, \sigma_{LT} \tag{18}$$

and their associated errors, respectively δ_1, δ_2 and δ_3, as well as the quantities:

$$S = \sum_i \frac{1}{\delta_i^2} \tag{19}$$
$$S_\varepsilon = \frac{\varepsilon_1}{\delta_1^2} + \frac{\varepsilon_4}{\delta_2^2} + \frac{\varepsilon_3}{\delta_3^2} \tag{20}$$
$$S_{\varepsilon\varepsilon} = \frac{\varepsilon_1^2}{\delta_1^2} + \frac{\varepsilon_4^2}{\delta_2^2} + \frac{\varepsilon_3^2}{\delta_3^2} \tag{21}$$
$$S_y = \sum_i \frac{y_i}{\delta_i^2} \tag{22}$$
$$S_{y\varepsilon} = \frac{y_1 \varepsilon_1}{\delta_1^2} + \frac{y_2 \varepsilon_4}{\delta_2^2} + \frac{y_3 \varepsilon_3}{\delta_3^2} \tag{23}$$

The χ^2 is then given by:

$$\chi^2 = [\frac{y_1 - \sigma_T - \sigma_{LTT}\, \varepsilon_1}{\delta_1}]^2 + [\frac{y_2 - \sigma_T - \sigma_{LTT}\, \varepsilon_4}{\delta_2}]^2 + [\frac{y_3 - \sigma_T - \sigma_{LTT}\, \varepsilon_3}{\delta_3}]^2 \tag{24}$$

and has to be minimized with respect to σ_T and σ_{LTT}. One obtains:

$$\sigma_T = \frac{S_{\varepsilon\varepsilon} S_y - S_\varepsilon S_{\varepsilon y}}{S S_{\varepsilon\varepsilon} - S_\varepsilon^2} \tag{25}$$
$$\sigma_{LTT} = \frac{S S_{\varepsilon y} - S_\varepsilon S_y}{S S_{\varepsilon\varepsilon} - S_\varepsilon^2} \tag{26}$$

with the errors:

$$\delta\sigma_T^2 = \sum_i \delta_i^2 \left(\frac{\partial \sigma_t}{\partial y_i}\right)^2 = \frac{S_{\varepsilon\varepsilon}}{S S_{\varepsilon\varepsilon} - S_\varepsilon^2} \tag{27}$$
$$\delta\sigma_{LTT}^2 = \sum_i \delta_i^2 \left(\frac{\partial \sigma_t}{\partial y_i}\right)^2 = \frac{S}{S S_{\varepsilon\varepsilon} - S_\varepsilon^2} \tag{28}$$

We need to fix a criteria in order to keep the most interesting time partitions. There are two possible criteria:

– taking the case for which $\frac{\delta\sigma_T}{\sigma_T}$ is minimum.
– taking the case for which $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ is minimum.

We have studied these two criteria and the results will be discussed in the next part.

4 Results

In order to perform an RLT separation with three ε values, we have to define a new kinematics with an ε_4 intermediate between ε_1 and ε_3 while keeping the same Q^2, p_m and ω. As ε is given by $[1 + 2\,q^2\,\mathrm{tg}(\theta_e/2)\,/\,Q^2]^{-1}$ and $q^2 = Q^2 + (E - E')^2$, we change both the beam energy E and the scattered electron angle θ_e.

In the case of the 2-body break up, results have been obtained both for the *central* quasi-perpendicular kinematics (see table 1 and 2) and the parallel ones (see table 3). A systematic error of 2% has been taken into account in the study and is part of the final errors given below.

Table 1 allows the comparison between the results from the EPERP program (with only two different ε values) to the results corresponding to a three ε values separation with $\varepsilon_4 = 0.649$ and for the two chosen minimization criteria. Table 2 is the same as table 1, but for $\varepsilon_4 = 0.567$.

For the quasi-perpendicular kinematics, the optimal T_4 obtained are unsignificant whichever relative uncertainty is being minimized ($\frac{\delta\sigma_T}{\sigma_T}$ or $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$). Indeed, in the best case, for $p_m = 0$ (which corresponds in fact to a parallel kinematics), $\frac{\delta\sigma_T}{\sigma_T}$ is decreased at most by 0.04$\%$ and $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ by 0.02$\%$. Similarly, for the parallel kinematics, the best improvements of $\frac{\delta\sigma_T}{\sigma_T}$ is equal to 0.08$\%$ and of $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ to 0.4$\%$.

These results show that it is not at all usefull to measure a fourth kinematics. One should also notice that T_4 is nearly independent both from the ε_4 value and from the chosen minimization criteria ($\frac{\delta\sigma_T}{\sigma_T}$ or $\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$).

5 Conclusions

The above study clearly demonstrated that an RLT separation with three ε values is not necessary and shall not decrease the error bars obtained with a two ε values separation. It is therefore useless to choose a fourth kinematics and to add a fourth cross section measurement.

References

[1] A. Saha, Computer Program HE3EEX.

[2] A. Saha, Computer Program EPERP.

[3] A. Saha and J. Mougey, Research Program at CEBAF, Vol 3, (1987)

[4] http://www.jlab.org/~kuss/e89044/e89044.html

Table Captions

Table 1: For quasi-perpendicular kinematics, comparison between the LT separation using two ε values and the method using three values with $\varepsilon_4 = 0.649$ and a beam energy of 2.045 GeV.

Table 2: Same as table 1 with $\varepsilon_4 = 0.567$ and a beam energy of 1.245 GeV.

Table 3: For parallel kinematics, same as table 1.

Separation		Kin.	Time					ERROR		
Num.	Method	p_m (GeV/c)	T_1 (Hours)	T_2 (Hours)	T_3 (Hours)	T_4 (Hours)	T_{lim} (Hours)	$\frac{\delta\sigma_{LT}}{\sigma_{LT}}$ (%)	$\frac{\delta\sigma_T}{\sigma_T}$ (%)	$\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ (%)
1	2 points for separation	0.000	3.40	3.40	13.21		20.00		2.35	11.36
	3 points for separation($\delta\sigma_T$)	0.000	1.19	1.19	14.56	3.06	20.00		2.31	11.34
	3 points for separation($\delta\sigma_{LTT}$)	0.000	3.10	3.10	12.74	1.07	20.00		2.31	11.33
2	2 points for separation	0.150	3.80	2.95	13.24		20.00	6.18	2.75	12.60
	3 points for separation($\delta\sigma_T$)	0.150	1.26	0.98	16.46	1.31	20.00	6.76	2.68	12.47
	3 points for separation($\delta\sigma_{LTT}$)	0.150	3.72	2.89	13.37	0.01	20.00	6.19	2.73	12.11
3	2 points for separation	0.300	4.12	2.81	13.07		20.00	13.13	12.96	47.20
	3 points for separation($\delta\sigma_T$)	0.300	1.46	1.00	17.54	0.00	20.00	24.54	11.68	54.97
	3 points for separation($\delta\sigma_{LTT}$)	0.300	3.94	2.68	13.37	0.01	20.00	13.41	12.80	46.40
4	2 points for separation	0.425	10.18	9.05	30.77		50.00	19.77	23.21	75.46
	3 points for separation($\delta\sigma_T$)	0.425	3.25	2.89	43.85	0.00	50.00	34.71	20.70	95.11
	3 points for separation($\delta\sigma_{LTT}$)	0.425	10.01	8.90	31.09	0.01	50.00	19.93	23.08	74.97
5	2 points for separation	0.550	37.34	51.76	110.89		200.00	27.97	32.79	98.69
	3 points for separation($\delta\sigma_T$)	0.550	14.72	20.40	164.71	0.17	200.00	44.44	28.94	124.27
	3 points for separation($\delta\sigma_{LTT}$)	0.550	38.08	52.78	109.06	0.08	200.00	27.71	33.13	100.32

Table 1: central perpendicular kinematics, ε_4=0.649, E_0= 2.045 GeV.

Separation		Kin.	Time					ERROR		
Num.	Method	p_m (GeV/c)	T_1 (Hours)	T_2 (Hours)	T_3 (Hours)	T_4 (Hours)	T_{lim} (Hours)	$\frac{\delta\sigma_{LT}}{\sigma_{LT}}$ (%)	$\frac{\delta\sigma_T}{\sigma_T}$ (%)	$\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ (%)
1	2 points for separation	0.000	3.40	3.40	13.21		20.00		2.35	11.36
	3 points for separation($\delta\sigma_T$)	0.000	1.17	1.17	12.68	4.98	20.00		2.26	11.36
	3 points for separation($\delta\sigma_{LTT}$)	0.000	3.35	3.35	12.91	0.39	20.00		2.27	11.35
2	2 points for separation	0.150	3.80	2.95	13.24		20.00	6.18	2.75	12.60
	3 points for separation($\delta\sigma_T$)	0.150	1.21	0.94	15.34	2.50	20.00	6.79	2.67	12.55
	3 points for separation($\delta\sigma_{LTT}$)	0.150	3.72	2.89	13.37	0.01	20.00	6.19	2.73	12.10
3	2 points for separation	0.300	4.12	2.81	13.07		20.00	13.13	12.96	47.20
	3 points for separation($\delta\sigma_T$)	0.300	1.46	1.00	17.54	0.00	20.00	21.55	11.68	54.97
	3 points for separation($\delta\sigma_{LTT}$)	0.300	3.94	2.68	13.37	0.01	20.00	13.41	12.80	46.40
4	2 points for separation	0.425	10.18	9.05	30.77		50.00	19.77	23.21	75.46
	3 points for separation($\delta\sigma_T$)	0.425	3.25	2.89	43.85	0.00	50.00	34.71	20.70	95.11
	3 points for separation($\delta\sigma_{LTT}$)	0.425	10.01	8.90	31.09	0.01	50.00	19.93	23.08	74.97
5	2 points for separation	0.550	37.34	51.76	110.89		200.00	27.97	32.79	98.69
	3 points for separation($\delta\sigma_T$)	0.550	14.72	20.40	164.71	0.17	200.00	44.45	28.94	124.28
	3 points for separation($\delta\sigma_{LTT}$)	0.550	38.08	52.78	109.06	0.08	200.00	27.71	33.12	100.32

Table 2: central perpendicular kinematics, ε_4=0.567, E_0=1.245 GeV.

Separation		Σ_4 Parameters		Time				ERROR	
Num.	Method	E_0 (GeV)	ε_4	T_1+T_2 (Hours)	T_3 (Hours)	T_4 (Hours)	T_{lim} (Hours)	$\frac{\delta\sigma_T}{\sigma_T}$ (%)	$\frac{\delta\sigma_{LTT}}{\sigma_{LTT}}$ (%)
1	2 points for separation			10.28	9.72		20.00	5.05	11.42
	3 points for separation($\delta\sigma_T$)	2.895	0.413	7.12	11.66	1.22	20.00	5.01	11.00
	3 points for separation($\delta\sigma_{LTT}$)	2.895	0.413	8.26	9.10	2.63	20.00	5.01	11.00
2	2 points for separation			10.30	9.69		20.00	3.60	26.25
	3 points for separation($\delta\sigma_T$)	2.895	0.310	6.64	9.96	3.40	20.00	3.52	26.24
	3 points for separation($\delta\sigma_{LTT}$)	2.895	0.310	10.10	9.22	0.67	20.00	3.53	26.23
3	2 points for separation			10.52	9.47		20.00	2.91	45.43
	3 points for separation($\delta\sigma_T$)	3.845	0.539	6.24	11.21	2.54	20.00	2.87	45.67
	3 points for separation($\delta\sigma_{LTT}$)	3.845	0.539	10.64	9.44	0.00	20.00	2.91	45.43
4	2 points for separation			8.06	11.96		20.00	7.14	15.76
	3 points for separation($\delta\sigma_T$)	1.945	0.819	4.62	15.38	0.00	20.00	6.84	16.69
	3 points for separation($\delta\sigma_{LTT}$)	1.945	0.819	7.94	12.05	0.00	20.00	7.12	15.76
5	2 points for separation			7.40	12.57		20.00	14.24	88.60
	3 points for separation($\delta\sigma_T$)	2.895	0.621	4.10	15.90	0.00	20.00	13.44	95.54
	3 points for separation($\delta\sigma_{LTT}$)	2.895	0.621	7.56	12.44	0.00	20.00	14.30	88.61
6	2 points for separation			10.90	9.10		20.00	10.69	46.49
	3 points for separation($\delta\sigma_T$)	2.895	0.904	7.08	12.92	0.00	20.00	10.07	49.77
	3 points for separation($\delta\sigma_{LTT}$)	2.895	0.904	10.94	9.05	0.00	20.00	10.70	46.49

Table 3: central parallel kinematics.