Chapter 2

Physics of Electron and Nucleon Scattering

2.1 Introduction to electron scattering

Scattering is the process in which a particle is deflected during an interaction. There are two main categories of scattering, elastic, where the particles are left intact and only the momentum changes, and inelastic, where the scattered partices are modified, excited, or destroyed. The differential scattering cross section is often written as the product of a cross section for scattering from a spin-0 point particle and a "form factor" which characterizes the structure of the target

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} f_{rec}|F(q^2, \theta_e)|^2 \ . \tag{2.1}$$

Here F is the form factor, q is the momentum transfer, the Mott cross section is that of a spin-$\frac{1}{2}$ electron scattered from a point-like particle, and f_{rec} is a kinematic factor (to be defined later). For a spin $\frac{1}{2}$ particle, $F(q^2, \theta_e)$ can be expressed in terms of the form factors G_E and G_M, which will be developed later[29].

The neutron is known to have no net charge. With high energy electrons, distances smaller tahn the size of a nucleon can be probed, allowing the structure to be described at different scales. Nonetheless, the neutron electric form factor is small compared to the other nucleon form factors. Various models exist that provide detailed descriptions of the nucleon over different kinematic ranges; specific

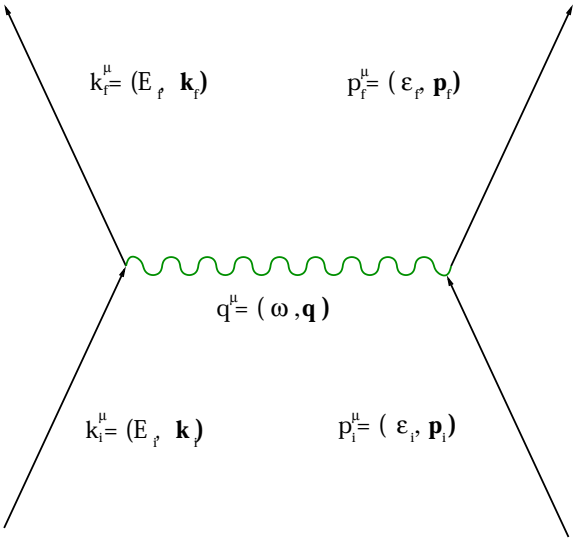

Figure 2.1: Feynman diagram for one interaction between an electron and a nucleon. The electron has four-momentum k^μ and the nucleon has four-momentum p^μ.

examples of models giving G_E^n at intermediate Q^2 will be given in more detail in section 3.1.

2.2 Definition of kinematic variables

The interactions of electrons and photons are well understood within the theory of quantum electrodynamics (QED). In this interaction, the properties of the incident and scattered electron and recoiling nucleon can be measured.

In the Feynman diagram in figure 2.1, the following kinematic variables are defined. The nucleon four-momentum is $p^\mu = (\epsilon, \mathbf{p})$, $k^\mu = (E, \mathbf{k})$ is the electron four-momentum, and $q^\mu = (\omega, \mathbf{q})$ is the virtual photon four-momentum. The four-momentum transferred to the nucleon is $q^2 = q_\mu q^\mu = \omega^2 - \mathbf{q}^2$, and is carried by the virtual photon. If evaluated in the laboratory frame, $\mathbf{p}_i = 0$ and $\epsilon_i = M_N$, where

M_N is the recoil nucleon mass. Since q^2 in electron scattering is space-like, Q^2 is best defined as $Q^2 = -q^2 = 4E_iE_f \sin^2 \frac{\theta_e}{2}$ (in the approximation $m_e = 0$), where θ_e is the angle the electron deviates from its initial direction. Q^2 is typically the quantity used to parametrize the form factors.

2.2.1 Current operator

The current of the electron in Figure 2.1 is arrived at from QED [29]

$$j^\mu(k_f, k_i) = \bar{u}(k_f)(-ie\gamma^\mu)u(k_i) \ . \tag{2.2}$$

Here γ^μ is the standard Dirac matrix and e is the electric charge. In this expression u is the Dirac spin $\frac{1}{2}$ spinor. For the spinor the following completeness relation is true

$$\sum_{s=1,2} u^{(s)}(p)\bar{u}^{(s)}(p) = \gamma^\mu p_\mu + m \ . \tag{2.3}$$

Similarly the current for a structureless, spin $\frac{1}{2}$ charged particle can be expressed as

$$J^\mu(p_f, p_i) = \bar{u}(p_f)(-ie\gamma^\mu)u(p_i) \ . \tag{2.4}$$

These provide the invariant amplitude $\mathcal{M}$ using the rules for Feynman diagrams. Following standard conventions, repeated indices are summed over:

$$i\mathcal{M} = J^\mu(p_f, p_i)\frac{-ig_{\mu\nu}}{Q^2}j^\nu(k_f, k_i) \ . \tag{2.5}$$

2.2.2 Dirac and Pauli form factors

The vertex factor for the nucleon-photon interaction is not exactly known, but rather parametrized in terms of form factors. These form factors must be

functions of q^2 since it is the only independent Lorentz scalar at the nucleon vertex in elastic scattering. Generally, the form factors can be constructed with any Lorentz-invariant constituent; γ^μ, γ^5, or $\sigma^{\mu\nu}$. However, the choices of $i\sigma^{\mu\nu}q_\nu$ and γ^μ are made to conserve parity. Other possible forms are restricted due to current and parity conservation. The two terms, γ^μ and $i\sigma^{\mu\nu}q_\nu$, give a current of

$$J^\mu(p_f, p_i) = \bar{u}(p_f)[-ie(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_N}F_2(q^2))]u(p_i) \ , \tag{2.6}$$

where F_1 and F_2 are known as the Dirac and Pauli form factors. They are normalized to give the proper charge and magnetic moment at $q^2 = 0$, so that

$$\begin{aligned} F_1^p(0) = 1, \ F_2^p(0) = \kappa_p \\ F_1^n(0) = 0, \ F_2^n(0) = \kappa_n \ . \end{aligned} \tag{2.7}$$

where κ is anomalous contribution to the nucleon's magnetic moment.

The Dirac and Pauli form factors are a parametrization of the electromagnetic structure in a different basis than the Sachs form factors. The Sachs form factors can be expressed in terms of F_1 and F_2 as [37]

$$G_E = F_1 - \tau F_2 \ , \ G_M = F_1 + F_2 \ , \tag{2.8}$$

where $\tau = \frac{Q^2}{4M_N^2}$.

2.3 Breit frame

As mentioned previously, the Sachs form factors can be related to the charge and magnetic moment distribution, in a particular reference frame called the Breit frame. No energy is transferred, only momentum ($\mathbf{p}_i = -\mathbf{p}_f$), therefore

$$Q^2 = \mathbf{q}_B^2 \ . \tag{2.9}$$

An important aspect of the Breit frame is that the Fourier transform of the electric form factor gives the charge density distribution, and that of the magnetic form factor gives the magnetic current density distribution.

The four components of the hadronic current in this frame are [29] [50]:

$$J^0 = ie2M_N\bar{u}_f(F_1 - \tau F_2)u_i = ie2M_N\bar{u}_f G_E u_i \ , \tag{2.10}$$

$$\overrightarrow{J} = -e\bar{u}_f(\overrightarrow{\sigma} \times \mathbf{q}_B)(F_1 + F_2)u_i = -e(\overrightarrow{\sigma} \times \mathbf{q}_B)\bar{u}_f G_M u_i \ . \tag{2.11}$$

Using this, Sachs[61] showed that G_E and G_M can be expressed as the Fourier transforms of the nucleon charge and magnetization densities:

$$G_E(Q^2) = \frac{4\pi}{Q}\int r dr \rho_{ch}(r)\sin Qr \tag{2.12}$$

$$G_M(Q^2) = \frac{4\pi}{Q}\int r dr \mu\rho_{mag}(r)\sin Qr. \tag{2.13}$$

At low Q, equation 2.12 can be expanded such that

$$G_E(Q^2) = 1 - \frac{1}{6}Q^2\langle r_E^2\rangle + \frac{Q^4}{120}\langle r_E^4\rangle + ... \tag{2.14}$$

From this we can see a simple formula for the charge radius of the neutron. At very low momentum transfer, the slope of the form factor with respect to Q^2 defines the mean square radius of the distribution[38],

$$< r_n^2 >= -6\frac{d(G_E^n)}{dQ^2}\Big|_{Q^2\to 0}. \tag{2.15}$$

This is obviously only true in the non-relativistic limit. Recently, Miller *et al.* have developed a formalism that allows a representation of the transverse charge and magnetization densities that are not reference frame dependent [**?**].

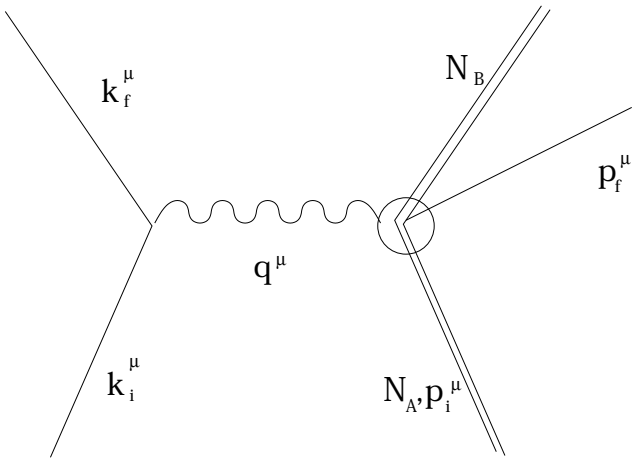

Figure 2.2: Feynman diagram for interaction between an electron and the nucleus. The electron has four-momentum k^μ and the ejected nucleon has momentum p_f^μ.

2.4 Quasi-elastic scattering

Rarely in scattering experiments is there an opportunity to scatter from a single nucleon. This is especially true of the neutron, which is short lived in its free state. There are many different approximations that can be used to put the nucleus in terms of elements that are better understood. Some of these are referenced in sections 5.5 and 5.7.1. While discussing all of them is out of the scope of this thesis, two which allow the final state nucleus to be understood in terms of a remainder nucleus and a scattered nucleon merit development. These are the impulse approximation (IA), and the plane wave impulse approximation (PWIA). Here and throughout much of this thesis, discussion will be formulated in terms of the one-photon exchange approximation.

2.4.1 Impulse approximation

In the impulse approximation the virtual (off-shell) photon interacts with a single nucleon. Afterward, the struck nucleon leaves the nucleus without any further interaction with the remaining "spectator" nucleons. This approximation can also be described as the assumption that the current of the nucleus is given by the sum of the currents of the individual nucleons, where the nucleons are treated as free particles[18].

The simplest nucleus that can be discussed in such terms is the deuteron. In the IA, the differential cross section for scattering from a neutron or proton embedded in the deuteron can be factorized. This allows the contributions of the neutron and proton to be separated. For an electron to scatter from a nucleon inside the deuteron, the contributions from the proton and the neutron can be separated

$$\frac{d\sigma_{ed}}{d\Omega} \approx \frac{d\sigma_{ep}}{d\Omega} + \frac{d\sigma_{en}}{d\Omega} \ . \tag{2.16}$$

This approximation, however, completely neglects nuclear binding and the inherent Fermi motion of the bound nucleons. It also neglects that the nucleon might re-scatter, the virtual photon might couple to a virtual meson exchanged between the nucleons, or the virtual photon might couple to an excited state of a nucleon (known as Isobar Configuration or IC). The quasi-elastically scattered nucleon interactions with the other "spectator" nucleons are known as final state interactions (FSI).

2.4.2 PWIA

A more detailed simple approximation is the Plane Wave Impulse Approximation (PWIA). This approximation is a quasi-free one, and is important in discussions of neutron double polarization experiments, where either the target or the recoiling nucleon is polarized. In PWIA the polarization of the recoiling nucleon or of the target is restricted to the scattering plane. The longitudinal component is parallel to the recoiling nucleon's momentum vector, and the transverse component is perpendicular[43]. In this approximation the initial and final state of the target nucleus are represented as products of plane waves for a nucleus and nucleon, where the nucleon is bound (with momentum $\mathbf{p}_i$). The nucleon absorbs the virtual photon, while the nucleus remains as a spectator. The nucleon then leaves the nucleus, which is assumed to be free after the absorption of the virtual photon, without any further interaction.

2.5 Polarization observables

A convenient coordinate system to work with observables when the beam, target, or recoiling nucleon are polarized is defined by[51]

$$\begin{aligned} \hat{z} &\parallel \mathbf{q}, \\ \hat{y} &\parallel \mathbf{k}_i \times \mathbf{k}_f, \\ \hat{x} &= \hat{y} \times \hat{z}. \end{aligned} \tag{2.17}$$

It is useful to refer to the polarization vector $\overrightarrow{P}$, which can serve as either the polarization of the nucleon inside the target in beam-target asymmetry mea-

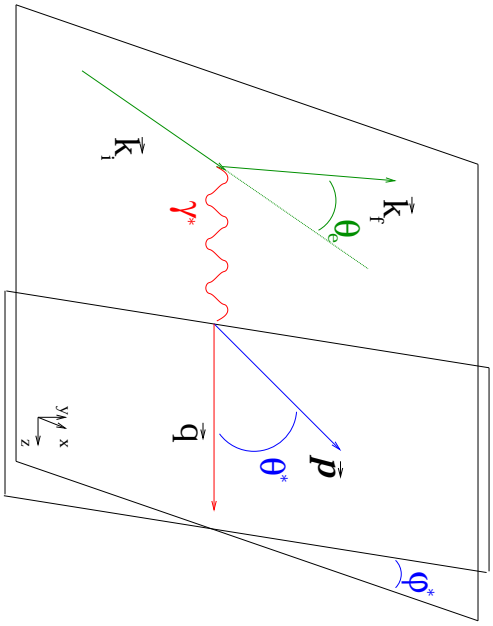

Figure 2.3: Polarization exchange in the one photon exchange approximation for polarized-electron, polarized nucleon scattering. θ and ϕ define the polarization of the recoiling nucleon.

surements, or the polarization of the ejected nucleon in beam-recoil polarization experiments.

$$\overrightarrow{P} = P(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}) \tag{2.18}$$

In the general case of parity conserving polarized scattering there are 18 independent response functions. Under the approximations of one photon exchange, PWIA, and parallel kinematics, only four terms survive. The cross sections of both a recoiling polarized nucleon and a polarized target share similar forms, and can be written in terms of products of kinematic quantities and response functions[37]. The kinematic variables (following the development in [37]) for these reduced terms are

$$V'_{TL} = (1-\tau)^{-1}\tan\frac{\theta_e}{2}\ , \tag{2.19}$$

$$V_T = \frac{1}{2(1-\tau)} \ , \tag{2.20}$$

$$V_L = \frac{1}{(1-\tau)^2} \ , \tag{2.21}$$

$$V'_{TT} = \sqrt{(1-\tau)^{-1} + \tan^2\frac{\theta_e}{2}} \tan\frac{\theta_e}{2} \ . \tag{2.22}$$

The response functions depend on the polarization of the recoiling nucleon and the target nucleus. These kinematic variables arise from the electron tensor in an alternate basis, where the coefficients refer to the nuclear response function with which the kinematic variable is associated ($_L$, $_{TL}$, $_T$ or $_{TT}$). The response functions arise from the nuclear response tensor ($W^{\mu\nu}$). The nuclear response tensor serves to parametrize the increased generality of the nucleus, where the same current doesn't exist in the initial and final states [29]. The four reduced response functions are $R'_{TT}(\hat{z})$, $R'_{LT}(\hat{x})$, R_L, and R_T, the first two of which reflect components associated with the polarized nature of the cross section. These subscripts $_L$ and $_T$ correspond to "longitudinal" and "transverse".

Two additional response functions exist for nuclei with more than one nucleon. These are R_{TT} and R_{TL} and only appear when the change in total angular momentum is greater than two units [21] [54].

2.5.1 Recoil polarization

From PWIA it can be shown that the unpolarized nuclear response tensor is symmetric, $W^{\mu\nu} = W^{\nu\mu}$. This means that the response tensor for recoil polarization is

$$W^{\mu\nu}(\hat{a}) = -W^{\nu\mu}(\hat{a}) = W^{\nu\mu}(-\hat{a}) \ . \tag{2.23}$$

In double-polarization experiments parallel kinematics are often used; under these kinematics the nucleon's final momentum is parallel to the momentum transfer. Also, at the electron beam energies in the experiments described here, only the longitudinal component of the electron polarization is relevant.

In the laboratory reference frame there are two response functions that describe the polarization (in PWIA), $R'_{TT}(\hat{z})$ and $R'_{LT}(\hat{x})$. These are the longitudinal and transverse response functions from which the transverse (P_x) and longitudinal (P_z) components of recoil polarization arise[37].

The general form of the nuclear structure tensor, with a polarized target or recoiling nucleon, can be written as[50]:

$$W_{\mu\nu} = J_\mu J^*_\nu = W^0_{\mu\nu} + W_{\mu\nu}(\overrightarrow{P}_f) + W_{\mu\nu}(\overrightarrow{P}_i) + W_{\mu\nu}(\overrightarrow{P}_i, \overrightarrow{P}_f) \ . \tag{2.24}$$

In recoil polarization experiments, typically only the final nucleon polarization is measured, and $W_{\mu\nu}$ reduces to (in the Briet frame):

$$W_{\mu\nu} = \frac{1}{2} Tr F_\mu F^\dagger_\nu \overrightarrow{\sigma} \cdot \overrightarrow{P} \ . \tag{2.25}$$

Here $F_0 = 2mG_E$ and $F_j = imG_M \vec{\sigma} \times \mathbf{q}_B \cdot \hat{j}$, where $j = x, y, z$. This gives $W_{\mu\nu}(P_x) \propto 2mG_E G_M$ and $W_{\mu\nu}(P_z) \propto G^2_M$.

As an example, this leads, after transformation back to the lab frame, to the following components of the recoil polarization of a free neutron [63][30]:

$$P_x = -P_e \frac{\sqrt{2\tau\epsilon(1-\epsilon)} G^n_E G^n_M}{\epsilon (G^n_E)^2 + \tau (G^n_M)^2} \ , \tag{2.26}$$

$$P_y = 0 \ , \tag{2.27}$$

$$P_z = P_e \frac{\tau\sqrt{1-\epsilon^2} (G^n_M)^2}{\epsilon (G^n_E)^2 + \tau (G^n_M)^2} \ , \tag{2.28}$$

where $\epsilon = [1 + 2(1+\tau)\tan^2\frac{\theta_e}{2}]^{-1}$, and P_e is the polarization of the electron beam.

2.5.2 Target polarization

The arguments and results are similar for a polarized target, where the detected nucleon is either unpolarized or not detected. Once more, in parallel kinematics, the one photon interaction approximation, and PWIA, only the $\hat{x}$ and $\hat{z}$ components of the target polarization vector remain.

In the response function formulation, and with spin-$\frac{1}{2}$ nucleons, to within a multiplicative constant, are a function of the components $F_L = (1+\tau)G_E$ and $F_T = \sqrt{2\tau(1+\tau)}G_M$. The general expression for a response function is

$$\sum_{j,j'} A_{j,j'}(k) F_j F_{j'} \tag{2.29}$$

where the $A_{j,j'}(k)$ has only two terms left for the polarized response, the F_T^2 and $F_L F_T$ terms. This formulation is developed in general for more complicated systems than elastic scattering with a polarized nucleon by Donnelly and Raskin [21] [54].

The response functions for the target are then[37]:

$$R_{LT}^T(\hat{x}) = 2\sqrt{\tau(1+\tau)}G_E G_M \ , \tag{2.30}$$

$$R_{TT}^T(\hat{z}) = 2\tau G_M^2 \ . \tag{2.31}$$

2.6 Cross section with unpolarized observables

The unpolarized differential cross section is dependent on the square of the scattering amplitude (spin averaged since it is unpolarized),

$$\frac{d\sigma}{d\Omega} = \frac{m_e^2}{4\pi^2}\frac{E_f}{E_i} f_{rec} \frac{1}{4} \sum |\mathcal{M}|^2, \tag{2.32}$$

where $f_{rec} = [1 + \frac{2E_i}{M}\sin^2\frac{\theta_e}{2}]^{-1}$. It is useful to define the Mott cross section, which corresonds to a relativistic electron scaterring from a structured target

$$\frac{d\sigma_{Mott}}{d\Omega} = \frac{\alpha^2 \cos^2\frac{\theta_e}{2}}{4E_i^2 \sin^4\frac{\theta_e}{2}}. \tag{2.33}$$

Here α is the fine structure constant.

Using the equations 2.2, 2.6, 2.32, and the Feynman rules the following relationship between the four-momenta, angles, and other elements of the electron and interacting fermion give the differential cross section. This relationship between the form factors and the cross section is also known as the Rosenbluth formula.

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha^2}{4E^2 \sin^4\frac{\theta_e}{2}}\right)\frac{E_f}{E_i}\left[\left(F_1^2 - \frac{q^2}{4M^2}F_2^2\right)\cos^2\frac{\theta_e}{2} - \frac{q^2}{2M^2}\left(F_1 - F_2\right)^2 \sin^2\frac{\theta_e}{2}\right] \tag{2.34}$$

Putting this in terms of the Sachs form factors gives[37]:

$$\frac{d\sigma}{d\Omega} = f_{rec}\left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta_e}{2}\right]. \tag{2.35}$$

This is much simpler than equation 2.34 and readily suggests a method of measurement of the form factors.

2.7 Cross section with polarized observables

The general form for the cross section when the beam of electrons is polarized and the polarization of the ejected particle is detected is:

$$\frac{d^6\sigma_{h,s}}{dE_f d\Omega_{k_f} d\epsilon_f d\Omega_{p_f}} = \sigma_{unpol}\frac{1}{2}(1 + \overrightarrow{P}_i \cdot \overrightarrow{\sigma} + h(A + \overrightarrow{P}_T \cdot \overrightarrow{\sigma})). \tag{2.36}$$

Here P_i is the induced polarization in the proton, h is the helicity of the incident electron, A is the beam analyzing power, s is the nucleon spin projection on σ, σ^{unpol} is the unpolarized cross section, and P_T is the polarization transfer coefficient[37]. In coplanar kinematics $A = 0$, $\mathbf{P}_i \cdot \overrightarrow{\sigma} = \mathbf{P}_i \cdot \hat{N}$, and $\mathbf{P}_T \cdot \hat{N} = 0$. Since $\mathbf{P}_i \cdot \hat{N} = 0$ in PWIA, two terms survive to provide the nucleon polarization, and from equations (**??**) and (**??**) an elegant measurement of G_E^n is possible.

This is similar in form to the cross section of polarized electrons on a polarized target where the recoil polarization is not measured:

$$\frac{d^6\sigma}{dE_f d\Omega_{k_f} d\epsilon_f d\Omega_{p_f}} = \sigma_{unpol}\frac{1}{2}(1 + \overrightarrow{P}_t \cdot \overrightarrow{A}_T + h(A + \overrightarrow{P}_t \cdot \overrightarrow{A}'_T)). \tag{2.37}$$

The new variables here are P_t which is the target polarization vector, A_T which is the target analyzing power, and A'_T which is the correlation vector[37].

Using a longitudinally polarized electron beam and a polarized target, the cross section can be written as:

$$\sigma^{pol} = \Sigma + h\Delta, \tag{2.38}$$

where Σ is the unpolarized cross section from equation 2.35, and Δ is the helicity dependent part[21]. Under the approximations of PWIA and one photon exchange,

and in the case of parallel kinematics and relativistic electron beam, the cross section simplifies to[37]:

$$\sigma^{pol} = \sigma_{Mott} f_{rec}[V_L R_L + V_T R_T + h P_{target}(V'_T R'_{TT}(\hat{z}) + V'_{TL} R'_{TL}(\hat{x}))]. \tag{2.39}$$

Using equations 2.19, 2.22, 2.31, 2.30, Δ is given by

$$\Delta = -2\sigma_{Mott} f_{rec} \tan\frac{\theta_e}{2}\sqrt{\frac{\tau}{1+\tau}}(\sqrt{\tau(1+(1+\tau)\tan^2\frac{\theta_e}{2})}\cos\theta G_M^2 + G_E G_M \sin\theta\cos\phi). \tag{2.40}$$