Crosscheck of the error propagation from target coordinate system to hall coordinate system

In optics analysis, the hall coordinate system (HCS) is defined by the intersection fo the electron beam and the vertical symmetry axis fo the target. Direction $\hat{z}$ points to downstream, $\hat{y}$ is vertically up and $\hat{x}$ is to the left facing downstream. The direction of a vector in this coordinate system can be expressed using two angle θ and ϕ, which are defined as normal spherical coordinates.

On the other side, the reconstructed target variables are defined in the target coordinate system (TCS) for convenience. The certral ray vertically passing through the central hole of the sieve slit defines the $\hat{z}_{\mathrm{tr}}$ axis. The $\hat{x}_{\mathrm{tr}}$ is vertically down facing the $\hat{z}_{\mathrm{tr}}$ axis and the $\hat{y}_{\mathrm{tr}}$ axis points to the right. A vector in TCS can be expressed using out-of-plane angle θ_{tr} and the in-plane angle ϕ_{tr}, which are given by $\mathrm{d}x_{\mathrm{tr}}/\mathrm{d}z_{\mathrm{tr}}$ and $\mathrm{d}y_{\mathrm{tr}}/\mathrm{d}z_{\mathrm{tr}}$.

The relation between (θ, ϕ) and $(\theta_{\mathrm{tr}}, \phi_{\mathrm{tr}})$ can be expressed as

$$\theta = \arccos(\cos\theta_{\mathrm{tr}}\cos(\theta_0 + \phi_{\mathrm{tr}})) \tag{1}$$

$$\phi = \arctan(-\frac{\tan\theta_{\mathrm{tr}}}{\sin(\theta_0 + \phi_{\mathrm{tr}})}) \tag{2}$$

where θ_0 is the spectrometer central angle. In our case, it is actually the central angle of the central ray of the sieve slit.

Thus we could do the calculation:

$$\begin{aligned} \mathrm{d}\theta = & \frac{-1}{\sqrt{1-\cos^2\theta_{\mathrm{tr}}\cos^2(\theta_0+\phi_{\mathrm{tr}})}}(-\sin\theta_{\mathrm{tr}})\cos(\theta_0+\phi_{\mathrm{tr}})\mathrm{d}\theta_{\mathrm{tr}} \\ & + \frac{-1}{\sqrt{1-\cos^2\theta_{\mathrm{tr}}\cos^2(\theta_0+\phi_{\mathrm{tr}})}}\cos\theta_{\mathrm{tr}}(-\sin(\theta_0+\phi_{\mathrm{tr}}))\mathrm{d}\phi_{\mathrm{tr}} \end{aligned} \tag{3}$$

$$\begin{aligned} \mathrm{d}\phi = & \frac{1}{1+\left(\frac{-\tan\theta_{\mathrm{tr}}}{\sin(\theta_0+\phi_{\mathrm{tr}})}\right)^2}\frac{-1}{\sin(\theta_0+\phi_{\mathrm{tr}})}\frac{1}{\cos^2\theta_{\mathrm{tr}}}\mathrm{d}\theta_{\mathrm{tr}} \\ & + \frac{1}{1+\left(\frac{-\tan\theta_{\mathrm{tr}}}{\sin(\theta_0+\phi_{\mathrm{tr}})}\right)^2}(-\tan\theta_{\mathrm{tr}})\left(-\frac{1}{\sin^2(\theta_0+\phi_{\mathrm{tr}})}\right)\cos(\theta_0+\phi_{\mathrm{tr}})\,\mathrm{d}\phi_{\mathrm{tr}} \end{aligned} \tag{4}$$

$$
\begin{aligned}
(\delta\theta)^2 &= \frac{\sin^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}{1-\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}(\delta\theta_{\rm tr})^2 + \frac{\cos^2\theta_{\rm tr}\sin^2(\theta_0+\phi_{\rm tr})}{1-\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}(\delta\phi_{\rm tr})^2 \\
&= \frac{\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}{1-\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}\left[\tan^2\theta_{\rm tr}(\delta\theta_{\rm tr})^2 + \tan^2(\theta_0+\phi_{\rm tr})(\delta\phi_{\rm tr})^2\right] \quad (5)
\end{aligned}
$$

$$
\begin{aligned}
(\delta\phi)^2 &= \left(\frac{1}{1+\left(\frac{-\tan\theta_{\rm tr}}{\sin(\theta_0+\phi_{\rm tr})}\right)^2}\right)^2 \\
&\quad \times\left[\left(\frac{1}{\cos^4\theta_{\rm tr}\sin^2(\theta_0+\phi_{\rm tr})}\right)(\delta\theta_{\rm tr})^2 + \left(\frac{\tan^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})}{\sin^4(\theta_0+\phi_{\rm tr})}\right)(\delta\phi_{\rm tr})^2\right] \\
&= \left(\frac{1}{1+\left(\frac{-\tan\theta_{\rm tr}}{\sin(\theta_0+\phi_{\rm tr})}\right)^2}\right)^2 \left(\frac{1}{\cos^2\theta_{\rm tr}\sin^2(\theta_0+\phi_{\rm tr})}\right)^2 \\
&\quad \times\left[\sin^2(\theta_0+\phi_{\rm tr})(\delta\theta_{\rm tr})^2 + \sin^2\theta_{\rm tr}\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})(\delta\phi_{\rm tr})^2\right] \\
&= \left(\frac{1}{\sin^2\theta_{\rm tr}+\cos^2\theta_{\rm tr}\sin^2(\theta_0+\phi_{\rm tr})}\right)^2 \\
&\quad \times\left[\sin^2(\theta_0+\phi_{\rm tr})(\delta\theta_{\rm tr})^2 + \sin^2\theta_{\rm tr}\cos^2\theta_{\rm tr}\cos^2(\theta_0+\phi_{\rm tr})(\delta\phi_{\rm tr})^2\right] \quad (6)
\end{aligned}
$$

These two equations (5 and 6) give us the uncertainty of (θ, ϕ) in HCS if we got the uncertainty of $(\theta_{\rm tr}, \phi_{\rm tr})$ in TCS from simulation.

In the simulation, target variables are calculated after the particles been generated. And after several steps to transport the particles forward and backward through the spectrometer and target field, one will get the reconstructed variables of this particle in TCS, with some uncertainty. If there is no target field, the uncertainty of the angle variables in HCS should be equal to the theoretical result (Eq. 5 and 6). But if the target field is on, the uncertainty of the target reaction point reconstruction will influence the final uncertainty since one will not be able to know the exact end point when particles are drifted in the target field. So if one complare the simulation result with the theoretical result, the contribution of the large uncertainty of the vertex z reconstuction to uncertainty of the scattering angle reconstruction can be distinguished.

There are two examples: For No field, $< \theta_{\rm tr} >=< \phi_{\rm tr} >= 0$, $\theta_0 \sim$

100 mrad

$$(\delta\theta)^2 = \frac{\cos^2\theta_0}{1-\cos^2\theta_0}\tan^2\theta_0(\delta\phi_{\rm tr})^2 = (\delta\phi_{\rm tr})^2 \tag{7}$$

$$(\delta\phi)^2 = \frac{1}{\sin^4\theta_0}\sin^2\theta_0(\delta\theta_{\rm tr})^2 = \frac{(\delta\theta_{\rm tr})^2}{\sin^2\theta_0} \tag{8}$$

from simulation, we got $\delta\theta_{\rm tr} = 1.02$, $\delta\phi_{\rm tr} = 0.842$. thus,

$$\begin{cases} \delta\theta_{\rm tr} = 1.02 \\ \delta\phi_{\rm tr} = 0.842 \end{cases} \Rightarrow \begin{cases} \delta\theta = 0.842 \\ \delta\phi = 10.2 \end{cases} \tag{9}$$

With a 5.0T target field and assume the beam energy is 2.253GeV, we will have $<\theta_{\rm tr}>\sim\theta_0\sim 100\,\rm mrad$, $<\phi_{\rm tr}>=0$, so

$$\begin{aligned} (\delta\theta)^2 &= \frac{\cos^2\theta_0\cos^2\theta_{\rm tr}}{1-\cos^2\theta_0\cos^2\theta_{\rm tr}}\left(\tan^2\theta_{\rm tr}(\delta\theta_{\rm tr})^2+\tan^2\theta_0(\delta\phi_{\rm tr})^2\right) \\ &= \frac{\cos^2\theta_0}{1+\cos^2\theta_0}\left((\delta\theta_{\rm tr})^2+(\delta\phi_{\rm tr})^2\right) \end{aligned} \tag{10}$$

$$\begin{aligned} (\delta\phi)^2 &= \left(\frac{1}{\sin^2\theta_{\rm tr}+\cos^2\theta_{\rm tr}\sin^2\theta_0}\right)^2\left(\sin^2\theta_0(\delta\theta_{\rm tr})^2+\sin^2\theta_{\rm tr}\cos^2\theta_{\rm tr}\cos^2\theta_0(\delta\phi_{\rm tr})^2\right) \\ &= \left(\frac{1}{1-\cos^4\theta_0}\right)^2\sin^2\theta_0\left((\delta\theta_{\rm tr})^2+\cos^4\theta_0(\delta\phi_{\rm tr})^2\right) \end{aligned} \tag{11}$$

use $\cos\theta_0 = 0.995$, we got

$$\begin{aligned} \delta\theta &= \sqrt{0.497((\delta\theta_{\rm tr})^2+(\delta\phi_{\rm tr})^2)} \\ \delta\phi &= \sqrt{25.3((\delta\theta_{\rm tr})^2+0.980(\delta\phi_{\rm tr})^2)} \end{aligned}$$

from simulation, we got $\delta\theta_{\rm tr} = 1.24$, $\delta\phi_{\rm tr} = 0.868$

$$\begin{cases} \delta\theta_{\rm tr} = 1.24 \\ \delta\phi_{\rm tr} = 0.868 \end{cases} \Rightarrow \begin{cases} \delta\theta = 1.14 \\ \delta\phi = 7.59 \end{cases} \tag{12}$$

But in the simulation, the uncertainty of $\delta\theta$ is 9.57 and the uncertainty of $\delta\phi$ is 76.2, which suggests that most of the uncertainty is from the bad resolution of the reaction point z reconstruction. We also compared the theoretical result with the simulation result under the assumption that we know the real reaction point z, the simulation result will fit the thoretical result under this condition.