

Yaw, pitch, and roll rotations

A 3D body can be rotated about three orthogonal axes, as shown in Figure [3.8](#). Borrowing aviation terminology, these rotations will be referred to as yaw, pitch, and roll:

1. A *yaw* is a counterclockwise rotation of α about the z -axis. The rotation matrix is given by

$$R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.39)$$

Note that the upper left entries of $R_z(\alpha)$ form a 2D rotation applied to the x and y coordinates, whereas the z coordinate remains constant.

2. A *pitch* is a counterclockwise rotation of β about the y -axis. The rotation matrix is given by

$$R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}. \quad (3.40)$$

3. A *roll* is a counterclockwise rotation of γ about the x -axis. The rotation matrix is given by

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{pmatrix}. \quad (3.41)$$

Each rotation matrix is a simple extension of the 2D rotation matrix, ([3.31](#)). For example, the yaw matrix, $R_z(\alpha)$, essentially performs a 2D rotation with respect to the x and y coordinates while leaving the z coordinate unchanged. Thus, the third row and third column of $R_z(\alpha)$ look like part of the identity matrix, while the upper right portion of $R_z(\alpha)$ looks like the 2D rotation matrix.

The yaw, pitch, and roll rotations can be used to place a 3D body in any orientation. A single rotation matrix can be formed by multiplying the yaw, pitch, and roll rotation matrices to obtain

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_x(\gamma) =$$

$$\begin{pmatrix} \cos \alpha \cos \beta & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \sin \alpha \cos \beta & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma \\ -\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma \end{pmatrix}. \quad (3.42)$$

It is important to note that $R(\alpha, \beta, \gamma)$ performs the roll first, then the pitch, and finally the yaw. If the order of these operations is changed, a different rotation matrix would result. Be careful when interpreting the rotations. Consider the final rotation, a yaw by α . Imagine sitting inside of a robot \mathcal{A} that looks like an aircraft. If $\beta = \gamma = 0$, then the yaw turns the plane in a way that feels like turning a car to the left.

However, for arbitrary values of β and γ , the final rotation axis will not be vertically aligned with the aircraft because the aircraft is left in an unusual orientation before α is applied. The yaw rotation occurs about the z -axis of the world frame, not the body frame of \mathcal{A} . Each time a new rotation matrix is introduced from the left, it has no concern for original body frame of \mathcal{A} . It simply rotates every point in \mathbb{R}^3 in terms of the world frame. Note that 3D rotations depend on three parameters, α , β , and γ , whereas 2D rotations depend only on a single parameter, θ . The primitives of the model can be transformed using $R(\alpha, \beta, \gamma)$, resulting in $\mathcal{A}(\alpha, \beta, \gamma)$.